International Journal of Theoretical Physics, Vol. 39, No. 6, 2000

Finitary Spacetime Sheaves

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Received February 14, 2000

The notion of finitary spacetime sheaves is introduced based on locally finite approximations of the continuous topology of a bounded region of a spacetime manifold. Finitary spacetime sheaves are seen to be sound mathematical models of approximations of continuous spacetime observables.

1. INTRODUCTION CUM PHYSICAL MOTIVATION

This paper associates with a finitary substitute F_n of a bounded region X of a continuous spacetime manifold M (Sorkin, 1991) a collection S_n of appropriately defined continuous functions on X, which, as a space on its own, is seen to be (locally) homeomorphic to F_n, thus, technically speaking, a sheaf over F_n (Bredon, 1967). This finitary spacetime sheaf is denoted by $S_n(F_n)$.

Then we consider an inverse system $\mathcal{F} = \langle F_n(\mathcal{U}_n)\rangle$ of finitary spacetime substitutes, derived from a net $\mathcal{L} = \langle \mathcal{U}_n\rangle$ of locally finite open covers of X as in Sorkin (1991), and the corresponding inverse system $\mathcal{N} = \langle S_n(F_n)\rangle$ of finitary spacetime sheaves associated with each element of $\mathcal{F}$. We show that as the elements F_n of $\mathcal{F}$ get more refined (in a sense to be defined), their corresponding sheaves S_n in $\mathcal{N}$ 'converge' to $S(X)$—the sheaf of (germs of) continuous functions on X.

The central physical idea we wish to model by the finitary spacetime sheaves S_n is 'locally finite approximations of the continuous spacetime observables on X'. In more detail, we intuit that as the continuous topology of X can be finitely (or coarsely) approximated by the finitary topologies F_n, so can the continuous maps on it, that constitute the sheaf $S(X)$, be effectively approximated by the finitary sheaves S_n. Since only the continuous (i.e., C^0)

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0020-7748/00/0600-1703$18.00/0

topological structure of X concerns us here, that is, we consider the continuous topology of spacetime to be its sole physically significant property, the physical spacetime observables that are of relevance are precisely the continuous functions on it. Then the maps in each finitary sheaf S_n over the locally finite substitute F_n of X represent sound coarse approximations of the continuous observables in $S(X)$ in the sense that, 'in the limit of infinite resolution' of the F_n into X, they effectively reproduce them. The latter may be formally written as $\lim_{n\to\infty} S_n(F_n) \equiv S_\infty(F_\infty) \simeq S(X)$ $[S_n(F_n) \in \mathcal{N}]$,[2] with an explanation of this limiting procedure pending.

The present paper is organized as follows: in Section 2, a brief reminder of finitary substitutes of a bounded region X of a continuous spacetime manifold M is given. We also allude, without proof, to the 'inverse limit' topological space F_∞ to which an appropriately defined 'inverse system' $\mathcal{H}$ of such substitutes derived from a net $\mathcal{L} = \langle \mathcal{U}_n \rangle$ of locally finite open covers of X converges 'at maximum resolution of X by $\mathcal{U}_\infty$'. This topological space is essentially homeomorphic to X. For detailed proofs the reader is referred to Sorkin (1991).

In Section 3, a space S that is locally homeomorphic to X is defined. This is the sheaf $S(X)$ of continuous functions on X.

In Section 4, finitary spacetime sheaves S_n of continuous functions on X associated with the finitary substitutes F_n of Section 2 are defined constructively.

Like their finitary 'domains' F_n, the finitary spacetime sheaves S_n also form an inverse system $\mathcal{N}$ having an inverse limit topological space $S_\infty(F_\infty)$ that is essentially homeomorphic to the sheaf $S(X)$ of Section 3. This limiting procedure is briefly presented in Section 5 by simulating in $\mathcal{N}$ Sorkin's proof that the inverse system $\mathcal{H}$ 'converges' to X. This is evidence of the soundness of the finitary sheaves as models of locally finite approximations of the continuous functions on X—the observables of the continuous spacetime topology of X.

In the concluding Section 6 we discuss the physical significance of the elements of finitary spacetime sheaves as the locally finite approximations of the continuous observables on the region X of the spacetime manifold M. In view of a recent definition of quantum causal sets (Raptis, 1999), partially motivated by the poset finitary substitutes of continuous spacetime topologies in Sorkin (1991), their quantum algebraic analogues in Raptis and Zapatrin (2000), and their causal relatives in Bombelli *et al.* (1987), we entertain the idea of a finitary spacetime sheaf of quantum causal sets, as well as the possibility that this structure be curved, and thus serve as a sound model of some sort of 'finitary quantum gravity'. However, the analytic development

[2] $\simeq$ denotes homeomorphic to.

of this possible application of finitary spacetime sheaves will have to be postponed for another paper (Mallios and Raptis, 2000).

2. FINITARY SPACETIME SUBSTITUTES REVISITED

In Sorkin (1991), with finite open covers $\mathcal{U}_n$ of a bounded region X of a continuous spacetime manifold M,[3] finite topological spaces F_n were associated by the following 'equivalence algorithm': two points x and y in X are said to be equivalent with respect to the open cover $\mathcal{U}_n$ (write $x \overset{\mathcal{U}_n}{\sim} y$) when $\forall U \in \mathcal{U}_n$: $x \in U \Leftrightarrow y \in U$.

Denoting by $\Lambda(X)$ the smallest open neighborhood in the subtopology $\mathcal{T}_n$ of X generated by the open sets U in $\mathcal{U}_n$,[4]

$$\Lambda(x) := \bigcap \{U \in \mathcal{U}_n | x \in U\}$$

one may alternatively define $x \overset{\mathcal{U}_n}{\sim} y$ as

$$(x \to y) \wedge (y \to x)$$

where $x \to y$ stands for $x \in \Lambda(y)$.

The quotient space $X/\overset{\mathcal{U}_n}{\sim} =: F(\mathcal{U}_n) \equiv F_n = \{[x]\}$ consisting of equivalence classes $[x]$ of points x in X relative to its finite open covering $\mathcal{U}_n$ is seen to be a T_0 topological space having the structure of a poset (Sorkin, 1991). With this T_0-quotienting of X by the equivalence relation $\overset{\mathcal{U}_n}{\sim}$ to F_n, that is, the substitution of X by its finitary approximation F_n relative to $\mathcal{U}_n$, a continuous function f_n from the subtopology $\mathcal{T}_n$ of X to F_n may be associated (Sorkin, 1991). f_n is continuous in the usual sense that open sets in F_n [5] are mapped by f_n^{-1} to open subsets of X in $\mathcal{T}_n$.

In Sorkin (1991), a net $\mathcal{L}$ of open covers $\{\mathcal{U}_n\}$ is also considered. For every pair $\mathcal{U}_i$ and $\mathcal{U}_j$ of locally finite open covers of X in $\mathcal{L}$, there is a 'finer' open cover $\mathcal{U}_k \in \mathcal{L}$ such that $\mathcal{U}_i, \mathcal{U}_j \subset \mathcal{T}_k$. With the net $\mathcal{L}$ and the f_n corresponding in its elements to the F_n, the inverse system $\mathcal{H} = \langle F_n \rangle$ of finitary substitutes of X may be derived. Here, too, $F_i \preceq F_k$ may be taken to stand for $F_i \subset \mathcal{T}_k$, and means that the open sets in the poset F_i are contained in the finitary poset topology $\mathcal{T}_k$ generated by the open sets in the poset F_k.

[3] A subset of a topological space is said to be bounded if its closure is compact. Our X in M is bounded in this sense.

[4] That is, $\mathcal{T}_n$ consists of arbitrary unions of finite intersections of the open sets U in $\mathcal{U}_n$.

[5] Defined with respect to the partial order relation $\to$ in F_n as the sets that can be obtained as unions of the following basic open sets: $\forall x \in F_n$: $\Lambda(x) := \{y \in F_n | y \to x\}$, where we have used the points x instead of the equivalence classes $[x]$ where they belong, for simplicity of notation.

For the pair i and k of indices above, the 'finer' relation may be denoted as $i \preceq k$ and it is seen to be a partial order between the locally finite open covers of X in $\mathcal{L}$ or their associated finitary substitutes in $\mathcal{K}$. Intuitively, the procedure of 'refinining' an open cover $\mathcal{U}_i$ to a finer $\mathcal{U}_j$ ($i \preceq j$) corresponds to adding more and finer or 'smaller' open subsets of X to the open cover $\mathcal{U}_i$ to obtain $\mathcal{U}_j$; thus, in a way, it represents the employment of a higher power of resolution in operations of determination of the topological structure of X.[6]

The net $\mathcal{L}$ of finite open covers of X provides the basis for the definition of $\mathcal{K}$ as an inverse system (Sorkin, 1991). For every pair $\mathcal{U}_i$ and $\mathcal{U}_j$ in $\mathcal{L}$ (or the corresponding pair of posets in $\mathcal{K}$), with $\mathcal{U}_i \preceq \mathcal{U}_j$ (or $F_i \preceq F_j$), one defines a map f_{ij}: $F_j \to F_i$, which is seen to be a unique continuous surjection.

The central result from Sorkin (1991) is that the inverse system $\mathcal{K}$ 'converges' to F_∞ at the (inverse) limit of maximum resolution (i.e., formally, as $n \to \infty$) of the finite open covers of X in $\mathcal{L}$. In particular, F_∞ is essentially f_∞-homeomorphic to X, so that, effectively, the maximally refined finitary substitute of X is topologically indistinguishable from (or equivalent to) it. This description of f_∞ as being 'essentially a homeomorphism' between X and F_∞ pertains to the fact, shown in Sorkin (1991), that X is f_∞-embedded in F_∞ as a dense subset. Then, Sorkin shows how to 'discard' from F_∞ the collection of its 'extra-points' y that are 'infinitely close' to those of $f_\infty(X)$, and thus establish f_∞ as a homeomorphism between $\hat{F}_\infty = F_\infty/\{y\}$ and X.[7]

The last result from Sorkin (1991) of interest to us here is that the tiling of F_∞ by the open sets in a particular covering $\mathcal{U}_i$ of X, symbolized as $f_{i\infty}^{-1}(x)$ ($\forall x \in F_i$),[8] becomes arbitrarily fine as $i \to \infty$. In other words, the open sets get 'smaller' and 'smaller' as the 'experimenter' employs higher power to resolve X into its points (i.e., ideally, to determine or localize individual spacetime events in X).

In the next section we present a space S that, similarly to F_∞, is (locally) homeomorphic to X.

3. *S*(*X*)—THE SHEAF OF CONTINUOUS FUNCTIONS OVER *X*

We want to organize the observables on the bounded spacetime region X of the continuous manifold M into a space S that, for all practical purposes, is topologically indistinguishable from (i.e., homeomorphic to) X. If we succeed in this, then the information about the topology of spacetime encoded

[6] We will return to this issue at the end of Section 5 and in Section 6.

[7] For the purposes of the present paper, we need not explain Sorkin's proof of the inverse limit in more detail. A brief outline of the basic steps of the proof is given in Section 5, where we argue that the $S_n(F_n)$ in $\mathcal{N}$ converge to $S(X)$.

[8] Where $f_{i\infty}$: $F_\infty \to F_i$ can be thought of as the continuous finite approximation of F_∞ by F_i.

in the points of X, which serve as the 'carriers of its topology' (Sorkin, 1991),[9] will be the same as that of our observations of them, at least locally,[10] and our description of the topological relations between spacetime events will be the same as that of the topological relations between our observations of them. Thus, the problem of localization of spacetime events will be effectively translated to the more operationally sound problem of localization of our observations of them.[11]

Now, there is such an organization of the continuous functions on X called a sheaf. Below, we first introduce the notion of a presheaf $P(X)$ of functions defined on the open sets of X, then we endow this presheaf with a topology that is locally equivalent (i.e., homeomorphic) to that of X. This process converts the presheaf $P(X)$ into the sheaf $S(X)$ of (germs of) continuous functions on X and is called 'sheafification'. Only the basic definitions from sheaf theory that will help us define finitary spacetime sheaves in the next section are given below. For a more detailed treatment of sheaves, the reader is referred to Bredon (1967).

A presheaf P on our region X of the continuous spacetime manifold M is an assignment to each open subset U of X of a set $P(U)$ and to each pair U, V of open subsets of X of a 'restriction map' $\rho_{U,V}$: $P(V) \to P(U)$ so that $\rho_{U,U} = id$ and $\rho_{U,V}\,\rho_{V,W} = \rho_{U,W}$, with id the identity map and $U \subset V \subset W \subset X$. One may think of the presheaf $P(X)$ as collections of functions defined on open subsets of X which ρ-reduce to one another when their respective domains of definition are nested by inclusion.[12]

To sheafify or 'topologize' the presheaf $P(X)$ to the sheaf $S(X)$, we embed each $P(U)$ in the presheaf to the collection $\Gamma(U, S)$ of sections of continuous functions on $U \subset X$ by the map σ_U: $P(U) \to \Gamma(U, S)$, that is, in some sense we select from $P(U)$ the continuous maps on U. σ_U commutes with the ρ-restrictions of open sets. Then, for every x in X we define the equivalence relation $\overset{x}{\sim}$ between the elements of the sets $P(U)$ and $P(V)$ $(U \cap V \neq \emptyset)$ as follows:

$$f \overset{x}{\sim} g, \quad (f \in P(U),\, g \in P(V))$$
$$\Leftrightarrow \rho_{W,U\cap V}(f) = \rho_{W,U\cap V}(g) \qquad (x \in W \subset U \cap V)$$

With the definition of $\overset{x}{\sim}$, one may define the stalk of the sheaf S over x as the following equivalence class of continuous functions at x:

[9] See Section 6 for a discussion of this physical role of the points of X.
[10] That is, 'about every point-event x in X'.
[11] Again, see Section 6 for more discussion on this.
[12] This nesting by inclusion of the open subsets of X is a partial order on the collection of all open subsets of X.

$$S_x := \bigcup \{P(U)|x \in U\}/\overset{x}{\sim} \equiv \overrightarrow{\lim}\{P(U)|x \in U\}$$

with $\overrightarrow{\lim}$ denoting 'direct limit'.[13] As a nontopologized set, the sheaf $S(X)$ may be expressed as a disjoint union (or direct sum) of its stalks $S(X) = \cup_{x \in X} S_X$.[14]

We may now endow S with the following topology $\mathcal{T}$: let f be a member of the presheaf $P(U)$ and x a point in U; then the germ of f at x, $[f]_x$, is the $\overset{x}{\sim}$-equivalence class of f. A basis for the topology of S consists of open sets of the following sort: $(x, [f]_x)$ $(x \in U)$.

Now, $\Gamma(U, S)$ above is the set of continuous sections of the sheaf $S(X)$ over its open subset U, that is, the set of continuous maps s: $U \rightarrow S$ such that, locally (i.e., pointwise in X), they map each point x in X to (an element $[f]_x$ of) the stalk S_x over it. Also, the selection σ_U in $\Gamma(U, S)$ of the map f in $P(U)$ above reduces locally to the germ of f at x, that is, $\sigma_U(f)(x) = [f]_x \in S_x$ $(x \in U)$. It follows that, as a topological space $\mathcal{T}(S)$, the sheaf $S(X)$ is generated by the (germs of) sections of the continuous maps on X, so that one can easily show that the 'projection map' π: $S(X) \rightarrow X$, given locally by $\pi(x, [f]_x) = x$, is a local homeomorphism; hence, the slogan that 'a sheaf is a local homeomorphism'. Equivalently, one can verify that every (germ of a) section of a continuous map in $\Gamma(U, S)$ is also such a local homeomorphism of X to $S(X)$,[15] so that for such a section $s \in \Gamma(U, S)$, the composition of s with π corresponds to the identity map id: $S(X)|_U \rightarrow S(X)|_U$ (i.e., $s \circ \pi \equiv id_{S(X)|_U} \Leftrightarrow s \equiv \pi^{-1}$), where $S(X)|_U$ is the restriction of the sheaf $S(X)$ to the open subset U of X.[16]

In the next section we define finitary spacetime sheaves $S_n(F_n)$ as the finitary substitutes of the continuous sheaf $S(X)$ in a way analogous to how

[13] This limit effectively yields the 'smallest' class of functions in the presheaf P over the 'finest' neighborhood of $x \in X$ (that is also included in both U and V). This direct limit effectively corresponds to maximum localization/resolution of $S(X)$ into its stalks S_x over the points x of X, and it is dual to the inverse limit procedure employed to resolve X into its points x in Sorkin (1991)—see the physical significance of finitary spacetime sheaves in Section 6.

[14] Thus, in some sense, the stalks S_x of $S(X)$ are its 'points', like the x in X—see last footnote and the physical significance of finitary spacetime sheaves in Section 6.

[15] Hence the equivalent slogan that 'a sheaf is its sections'.

[16] We may add here that usually the stalks $S_x \in S(X)$ are assumed to have some kind of algebraic structure, so that the algebraic operations that define it respect the 'horizontal continuity' of the base space X (i.e., they satisfy 'compatibility conditions' with the topology of the underlying space X). It must be noted that the continuous functions on X form indeed an algebra, the prototype being $C^0(X, \mathbb{C})$—the algebra of complex-valued continuous functions on X. It is tacitly assumed that the operations in $C^0(X, \mathbb{C})$ respect the base topology in a sheaf of such algebras over X. In the present paper we are not interested in the algebraic structure of the stalks of the (finitary) sheaves. In Section 6, however, we mention finitary spacetime sheaves whose stalks have some specific algebraic structure of special importance to the physical situation that they are employed to model.

the F_n were seen to be the locally finite poset substitutes of the continuous topological space X in Sorkin (1991) and Section 2.

4. CONSTRUCTIVE DEFINITION OF $S_n(F_n)$

A point in X (i.e., a spacetime event) corresponds to an ideal determination of location in spacetime—an ideal measurement of the locus of an event. A more pragmatic and operationally sound model of spacetime measurements, one taking into account their actual 'roughness' or 'approximate character', or even the 'fuzziness' due to the uncontrollable perturbations that such realistic acts of measurement inflict on spacetime, is that they at least determine open sets in X (Sorkin, 1995; Raptis and Zapatrin, 2000). Since points in X serve as the 'carriers of the spacetime topology' (Sorkin, 1991; see below, Section 6), the aforementioned pragmatic determinations of events by open sets of X effectively correspond to 'approximations of its continuous topology'. Thus, the latter may be thought of as the ideal limit-topology an experimenter determines at his (also ideal!) maximum power of resolution of spacetime into its finest/'smallest' open sets containing its points. This was essentially the moral of Section 2.

Now our main physical motivation for defining finitary spacetime sheaves S_n for every finitary approximation F_n of the continuous topology of X is to similarly approximate the sheaf S of continuous functions on X, and effectively recover it at the limit of maximum resolution into its 'ultralocal elements' that in the previous section were seen to be its stalks S_x over the points x in its base space X. As we saw in the last section, the stalks of S over the points of X consist of the germs of sections of continuous functions on X. Thus, the main idea is to define our finitary sheaves $S_n(F_n)$ in such a way that their stalks consist of 'gross observables' that are continuous over the 'rough' open subsets of a finite cover $\mathcal{U}_n$ of X and that, in the limit of maximum resolution of the $\mathcal{U}_n$ to F_∞ as in Section 2, they reduce to the S_x of $S(X)$. This (inverse) limit will be the subject of the next section.

The finitary spacetime sheaf S_n associated with the locally finite open cover $\mathcal{U}_n$ of X is defined as follows: take a finitary open cover $\mathcal{U}_n$ of X. Recall from Section 2 that $\overset{\mathcal{U}_n}{\sim}$ denotes the following equivalence relation between points x and y of X:

$$x \overset{\mathcal{U}_n}{\sim} y \Leftrightarrow (x \to \Lambda(y)) \wedge (y \to \Lambda(x)); \qquad \Lambda(X) := \bigcap\{U \in \mathcal{U}_n |\, x \in U\}$$

according to which the poset finitary substitute F_n of X with respect to its locally finite open cover $\mathcal{U}_n$ is defined by the quotient $F_n := X/\overset{\mathcal{U}_n}{\sim}$ and consists of equivalence classes $[x]$ of points $x \in X$. Then, force the following 'collapse' equivalence relation between the stalks $S_x = \{[f]_x\}$ of $S(X)$ over the x in X, induced by the $\overset{\mathcal{U}_n}{\sim}$ equivalence relation between them, as

$$x \overset{\mathcal{U}_n}{\sim} y \Rightarrow [f]_x \overset{\mathcal{U}_n}{\equiv} [f]_y$$

where we have effectively identified the germs of continuous functions in the stalks S_x and S_y of $S(X)$ over the $\overset{\mathcal{U}_n}{\sim}$-equivalent points x and y in X.[17] This $\overset{\mathcal{U}_n}{\equiv}$-identification (equivalence relation) of the stalks of $S(X)$ has the following physical interpretation: for point-events in X that are $\overset{\mathcal{U}_n}{\sim}$-indistinguishable at the power of resolution employed to analyze it by $\mathcal{U}_n$, one may use any of the germs of the continuous observables from $S(X)$ residing in the stalks over them in order to describe their 'local topological relations'.[18]

Then, like the F_n, we define the corresponding finitary sheaf $S_n(F_n)$ as the following equivalence class of stalks in $S(X)$:

$$S_n(F_n) := S(X)/\overset{\mathcal{U}_n}{\equiv} = \bigcup_{[x]\in F_n} S_{[x]}$$

It is plain to see that the finitary sheaves $S_n(F_n)$ inherit the poset T_0-topology of their corresponding finitary substitutes F_n as follows:

$$[x] \to [y] \in F_n \Rightarrow [f]_{[x]} \to [f]_{[y]} \in S_n(F_n)$$

with the partial order in $S_n(F_n)$ pending a bit of further explanation. This explanation can be drawn straightforwardly by considering the following 'commutative diagram':

$$\begin{array}{ccc} X & \xrightarrow{f_n} & F_n \\ \pi^{-1}\downarrow s & & s_n\downarrow \pi_n^{-1} \\ S & \xrightarrow{\hat{f}_n} & S_n \end{array}$$

where $s_n \equiv \pi_n^{-1}$ is the 'local homeomorphism' from F_n to S_n that we are searching for. The diagram shows that, as a map $s_n = \hat{f}_n \circ \pi^{-1} \circ f_n^{-1}$, where f_n^{-1} is the inverse of the bijective correspondence between the smallest open subsets $\{\Lambda(x)\}$ containing the points of X with respect to its locally finite open covering $\mathcal{U}_n$ and the $\overset{\mathcal{U}_n}{\sim}$-equivalence classes of points of X in F_n,[19]

[17] Note that all our definitions are implemented 'pointwise in X'. As noted earlier, we hold to the primitive intuition that the points of X are 'the carriers of its topology'—the basic tenet of point set topology (see the physical motivation in Section 6 for a 'physical justification' of this primitive intuition).

[18] Again, see Section 6 for more on this.

[19] Defined in Section 2 as the map f_n: $\mathcal{T}_n \to F_n$. f_n as a map from X to F_n is a continuous surjection, but as a map from $\mathcal{T}_n$ to F_n it is a homeomorphism (i.e., a bijective continuous map whose inverse is also continuous).

$\pi^{-1} \equiv s$ is the local homeomorphism from X to S as defined in the previous section, which, in terms of the corresponding smallest open neighborhoods $\Lambda(x)$ of points x in X with respect to $\mathcal{U}_n$, reads s: $\Lambda(x) \to \Gamma(\Lambda(x), S)$, and $\hat{f}_n$ is the one-to-one map defined by the equivalence relation $\overset{\mathcal{U}_n}{\equiv}$ between the stalks of S with respect to $\mathcal{U}_n$. Thus, as the partial order $x \to y$ (or $[x] \to [y]$) in F_n stands (by definition) for $x \in \Lambda(y)$,[20] $[f]_{[x]} \to [f]_{[y]}$ in S_n is simply interpreted as the following set-theoretic inclusion in S: $[f]_{[x]} \in \{[f]_{\Lambda([y])}\} = \pi^{-1}(\Lambda(y)) = S_{\Lambda(y)} = \cup_{x \in \Lambda(y)} S_x$—the restriction of $S(X)$ at $\Lambda(y) \subset X$. All in all, s_n: $F_n \to S_n$ is a local homeomorphism because, by construction, it is one-to-one and (locally) preserves the T_0 order-topology of both F_n and S_n.

This completes the definition of the finitary sheaf S_n over the locally finite poset substitute F_n of X. In the next section we argue that, as the inverse system $\mathcal{K} = \langle F_n \rangle$ converges to X, so its derivative $\mathcal{N} = \langle S_n(F_n) \rangle$ converges to $S(X)$.

5. THE INVERSE LIMIT $S(X)$ OF $\langle S_n(F_n) \rangle$

In this section we present briefly the basic definitions and follow in some detail the main steps in Sorkin's proof of the 'convergence' of the inverse system $\mathcal{K}$ of finitary substitutes F_n and the maps f_{ij} between them to a space S_∞ that is essentially homeomorphic to X, and apply it for the proof of a similar 'convergence' of the inverse system $\mathcal{N}$ of the finitary sheaves of the previous section and certain continuous surjections $\tilde{f}_{ij}$ between them, to a limit space $S_\infty(F_\infty)$ that is effectively homeomorphic to $S(X)$. The essential point is that, since our constructive definition of the finitary sheaves S_n in the previous section followed precisely, pointwise in X, the steps of the constructive definition of their respective domains F_n in Section 2, so that both F_n and S_n have the same poset-topology, one expects the proof of the convergence of $\mathcal{N}$ to $S(X)$ to be effectively the same as that of $\mathcal{K}$ to X given in Sorkin (1991). Thus, our proof of the latter convergence only highlights the important definitions and proof steps given by Sorkin (1991), and we refer the reader to it for more details.

The first thing from Sorkin (1991) that we mention is that 'convergence' of $\mathcal{K}$ does not pertain to the usual notion of a 'limit' for its terms, because for that a topology on the set of all topologies on X would have to exist, which is not the case. Rather, $\mathcal{K}$ is defined as an inverse system possessing an inverse limit.

The terms in $\mathcal{K}$ are the finitary substitutes of X corresponding to the net $\mathcal{L}$ of locally finite open covers of X, together with unique continuous

[20] And it literally stands for the convergence of the constant sequence x to y (Sorkin, 1991).

surjections f_{ij}: $F_j \to F_i$ $(i \preceq j)$[21] between them. Thus, the inverse system $\mathcal{N}$ is defined in the same way as $\mathcal{H}$, though the continuous surjections between the finitary sheaves in it are now denoted by $\tilde{f}_{ij}$ $(i \preceq j)$.[22]

As the inverse system $\mathcal{H} = \langle F_j, f_{ij} \rangle$ is seen to possess an inverse limit $(F_\infty, f_{i\infty})$ as $j \to \infty$, with F_∞ a T_0 topological space like all the finite terms in $\mathcal{H}$ (Sorkin, 1991),[23] so does the inverse system $\mathcal{N} = \langle S_j(F_j), \tilde{f}_{ij} \rangle$. The limit sheaf space $S_\infty(F_\infty)$ is a T_0 topological space, since all the finite terms $S_n(F_n)$ in $\mathcal{N}$ are T_0 posets.

The reader is referred to a series of lemmas in Sorkin (1991) that establish that the inverse limit space F_∞ of the inverse system $\mathcal{H}$ is a non-Hausdorff space[24] that contains X as a dense subset. Since, as we noted earlier, the definitions and constructions for the terms in the inverse system $\mathcal{N}$ are identical, being implemented pointwise in X, with those in $\mathcal{H}$ of Sorkin (1991), we directly infer that the lemmas mentioned above also hold in our scheme, so that we may quote directly their result: the inverse limit $S_\infty(F_\infty)$ of $\mathcal{N}$ is a non-Hausdorff space that contains $S(X)$ as a dense subset. The latter means essentially that for every stalk S_y over $y \in F_\infty$ in the limit sheaf S_∞ over F_∞, there is a stalk S_x in $f_\infty(S(X))$ 'infinitely close' to it.[25]

Next, we mention a further lemma in Sorkin (1991) that establishes that if X is T_1 and the previous lemmas for the denseness of X in F_∞ also hold, then $f_\infty(X)$ constitute the points of F_∞ that are closed in its topology. We apply it to our situation and state that the image set $f_\infty(S(X))$ in $S_\infty(F_\infty)$ consists of the closed stalks in the latter's topology. We refer again to Sorkin (1991) to verify that F_∞, and *in extenso* our $S_\infty(F_\infty)$, is non-Hausdorff; moreover, one can 'delete' its extra points, and thus render f_∞ a homeomorphism between $S(X)$ and $S_\infty(F_\infty)$.

Finally, we mention the usefulness for the physical interpretation of the finitary spacetime sheaves $S_n(F_n)$ as finite approximations of $S(X)$ in the next section (the sheaf-theoretic analogue of Sorkin's proof) that the open subsets $\tilde{f}_{i\infty}^{-1}(S_{[x]})$ $(S_{[x]} \in S_i(F_i))$ tiling $S_\infty(F_\infty)$ become arbitrarily fine, or 'small' as $j \to \infty$ (Sorkin, 1991; see above, end of Section 2). Thus, refining the finitary

[21] The uniqueness of the continuous f_{ij} follows from the universal mapping theorem for T_0 topological spaces and the assumption that $i \preceq j$ (Sorkin, 1991).

[22] The 'finer' relation $S_i \preceq S_j$ means in this case that the poset topology $\mathcal{T}(S_i)$ is a subtopology of $\mathcal{T}(S_j)$, as for their corresponding finitary substitutes.

[23] Again, the proof is via the universal mapping property of the f_{ij}.

[24] A topological space X is said to be Hausdorff, or satisfying the T_2 axiom of separation of point-set topology, if for every pair of distinct points x and y in it, there exist open neighborhoods $N(x)$ and $N(y)$ about them such that $N(x) \cap N(y) = \emptyset$.

[25] One should see the lemmas in Sorkin (1991) that establish this 'infinite closeness' relation between points in F_∞ and $f_\infty(X)$, and convince oneself that they also hold, pointwise, in our sheaf-theoretic scheme as well.

sheaves in $\mathcal{N}$ effectively amounts to better localizations or approximations of the observables residing in $S_x \in S(X)$ (see next section).

6. PHYSICAL SIGNIFICANCE AND A FUTURE APPLICATION OF THE S_n

If we consider a net $\mathcal{L} = (\{\mathcal{U}_n\}, \prec)$ of locally finite open covers of X, with $\mathcal{U}_i \prec \mathcal{U}_j$ denoting the relation 'finer' between its elements as defined in Section 2,[26] then as shown in Sorkin (1991) and briefly discussed above, "in the limit of infinite resolution of X into its 'smallest open' neighborhoods (in $\mathcal{U}_\infty$) about its points, X is recovered up to homeomorphism," that is, formally, $F_\infty(\mathcal{U}_\infty) := \lim_{n\to\infty} F_n(\mathcal{U}_n)$ is homeomorphic to X. Thus, finitary substitutes are regarded as sound finite approximations of continuous topologies (Sorkin, 1991), whereby a 'rough'[27] determination of a point in X is modeled after an open neighborhood about it (Sorkin, 1991, 1995).

Now, the transcription of the problem 'F_n as approximations of X' to '$S_n(F_n)$ as approximations of $S(X)$' that is the essence of the present paper changes focus from 'approximate localization/local determination (measurement) of points in X' to 'approximate localization/local determination (measurement) of continuous functions over X', and thus from a physical point of view, when X is taken to be a bounded region of the spacetime manifold, from 'localization of events' to 'localization of observables of events'. In this paper, as mentioned earlier, the continuous topology of X is regarded as its sole physically significant property 'carried by its points', and thus the continuous functions on it adequately qualify as 'spacetime observables of events'. This attribute of points as 'carrying the topology of X' can be realized by requiring that every physical space X is a T_0 topological space (Sorkin, 1991). The relation $\stackrel{\mathcal{U}_n}{\sim}$ between events in X is physically interpreted as 'indistinguishability of events at the finite power of resolution of X corresponding to $\mathcal{U}_n$', and $\stackrel{\mathcal{U}_n}{\equiv}$ between the observables residing in the stalks of $S(X)$ as 'indistinguishability of the observables of events at the finite power of resolution of $S(X)$ corresponding to $\mathcal{U}_n$'.

The posets corresponding to the locally finite substitutes F_n of X are known to have an equivalent (i.e., functorial) representation as simplicial complexes obtained from the nerves of the covering $\mathcal{U}_n$ (Alexandrov, 1956; Raptis and Zapatrin, 2000), which, in turn, are categorically equivalent to incidence Rota algebras associated with them (Raptis and Zapatrin, 2000;

[26] Roughly, $\mathcal{U}_j$ has more and 'smaller' open subsets of X than $\mathcal{U}_i$.
[27] 'Fuzzy', 'blurry', or 'foamy' may be regarded as alternative synonyms to 'rough'.

Zapatrin, 2000).[28] Observables, too, are in a certain sense dual to events.[29] In Raptis and Zapatrin (2000) it was found that the Rota incidence algebra R_n associated with a particular finitary substitute F_n is also a discrete differential manifold in the sense of Dimakis and Müller-Hoissen (1999). Thus, not only the reticular analogues of the continuous (C^0) functions on X are encoded in a finitary sheaf R_n of Rota algebras over the F_n, but also a discrete version of the smooth (C^∞) ones.[30] Hence, the conjecture is that at the maximum resolution $\mathcal{U}_\infty$ of a bounded region X of a smooth spacetime manifold M, the Rota algebras R_n associated with the locally finite posets F_n, R_n (F_n), are expected to 'yield' (X, ∂, Ω)[31]—the flat sheaf of sections of smooth differential forms over X (i.e., the smooth spacetime observables).[32] Also, a quantum interpretation has been given to the Rota incidence algebras R_n associated with the finitary substitutes F_n of X (Raptis and Zapatrin, 2000). Accordingly, the limit space of a system of $R_n(F_n)$ is (X, ∂, Ω)—the classical smooth spacetime manifold with differential forms attached, and it was interpreted there as Bohr's correspondence limit structure of a quantal substratum of finitary incidence algebras.

In Raptis (1999), a causal interpretation to the quantal incidence algebras $R_n(F_n)$ of Raptis and Zapatrin (2000) was given. The resulting structures were called 'quantum causal sets'—a quantal version of the causal sets of Bombelli *et al.* (1987). It follows from the discussion above that a finitary sheaf of quantum causal sets may be studied as a quantal and locally finite substitute of the causal relations between events in a bounded

[28] That is, the category of finitary posets/order morphisms is functorial to that of simplicial complexes/simplicial mappings, which, in turn, is 'antifunctorial' to that of Rota algebras/Rota homomorphisms. The latter means that, since the Rota incidence algebras associated with finitary posets are objects dual to them (Raptis and Zapatrin, 2000), there is a contravariant functor between their respective categories. Thus, while finitary posets constitute the inverse system $\mathcal{H}$ in Sorkin (1991) and above, their associated incidence algebras may be organized into a 'direct system' having a direct limit, much like the finitary sheaves $S_n(F_n)$ above were seen to have a direct limit space isomorphic to the stalk S_x of $S(X)$.

[29] Intuitively they are dual, for the pairing of an observable f with a point-event x produces a measurable number $f(x)$ (i.e., the value of the 'field' f at the 'test-event' x). Thus, in some sense, x is like a state of X, while the action of f on it, $f(x)$, is some sort of measurement of the property f of X at x (here, its continuous topology). Evidence for this duality is the mathematical duality of the notions of inverse and direct limit by which the local (pointlike) elements of X and $S(X)$ (i.e., x and S_x) were defined above, respectively. See also previous footnote.

[30] Although our study in the present paper concentrates solely on the continuous (i.e., C^0) structure of spacetime, not its differential.

[31] Ω being the module of differential forms over the algebra of C^∞-smooth functions on X, i.e., $\Omega := \Omega^0 (\equiv C^\infty(X)) \oplus \Omega^1 \oplus \Omega^2 \oplus \ldots$.

[32] In Mallios (1998), this structure is called the 'smooth and flat differential triad'. The stalks of this sheaf are isomorphs of Ω and the Kähler–Cartan differential ∂ effects (stalkwise) vector sheaf morphisms of the following sort: $\partial: (X, \Omega^n) \to (X, \Omega^{n+1})$, where (X, Ω^n) is the vector subsheaf of (X, ∂, Ω) having as stalks isomorphs of the vector space Ω^n of n-forms, which is a vector subspace of Ω.

region of a smooth Lorentzian spacetime manifold. As in Dimakis and Müller-Hoissen (1999), a Riemannian metric connection was studied on a discrete differential manifold, so we should be able to define a nonflat pseudo-Riemannian connection on the finitary sheaf of quantum causal sets by using powerful sheaf-theoretic results from Mallios, (1998).[33] It is expected that in the Bohr correspondence limit[34] of an inverse system of finitary sheaves of quantum causal sets, the smooth (region of a) Lorentzian manifold, together with the smooth fields and a nonflat pseudo-Riemannian metric connection D on it,[35] will emerge. Then, the underlying finitary sheaves of quantum causal sets may be regarded as sound models of a reticular and quantal version of gravity.[36] This project, however, is still under development (Mallios and Raptis, 2000).

We conclude by discussing the 'general moral' of the present paper. By considering the finitary sheaves approximating $S(X)$ rather than directly the finitary spaces approximating X, we regard our observations of spacetime events as being more fundamental than the events themselves. This is the main lesson for physics to learn by applying differential geometry in the inherently algebraic language of sheaf theory, namely, that the physically significant concepts are less those about the 'geometrical' background spacetime X and more those about our observations of this background which are organized into sheaves (of algebras) over X (Mallios, 1998). This general principle that underlies the abstract differential geometry via vector sheaves theory developed in Mallios (1998) is well in accord with the general philosophy of quantum theory holding that inert, background, geometrical 'state spaces' such as spacetime 'dissolve away', so that what remains and is of physical significance, the 'physically real' so to speak, is (the algebraic mechanism of) our own actions of observing 'it' (Finkelstein, 1996).[37]

[33] These results were obtained for a paracompact and Hausdorff topological base space X of the vector sheaves considered there. A space that is paracompact is akin to one that is finitary in our and Sorkin's (1991) sense of the latter denomination (i.e., that it admits a locally finite open cover). Our X, which is such a finitary topological space, was assumed to be bounded (i.e., having compact closure), and in the usual topological parlance it is called 'relatively compact'. X was also seen to be T_1, but not T_2 (i.e., not Hausdorff). If we relax 'paracompactness' to 'relative compactness' and T_2 to T_1 for X, the essential results of abstract differential geometry via vector sheaves still apply to our case (A. Mallios, private communication).

[34] In the sense of Raptis and Zapatrin (2000).

[35] Interpreted as the classical gravitational potential.

[36] A 'finitary quantum gravity' so to speak.

[37] To parallel Saunders Mac Lane's mathematical motto, 'Every good function is a section of a sheaf' (Mac lane, 1986), in physics: 'Every physically significant action is a section of the sheaf of our observations of the system in focus'.

ACKNOWLEDGMENTS

The unceasing support—spiritual, moral, technical, and material—of Anastasios Mallios is wholeheartedly acknowledged. Exchanges on finitary spacetime substitutes during a year of collaboration with Roman Zapatrin are also greatly appreciated. Finally, this paper was written with the help of a Postdoctoral Research Fellowship in Mathematics from the University of Pretoria.

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